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Optimization approach for Harmonic Regime in Electromagnetics

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Abstract: We deal with new optimization-control approaches for the approximation of harmonic regimes in electromagnetics. This framework is robust in the sense that it can easily be extended to variable dialectric media with no specific regularity on the conductivity and permitivity coefficients which could also be time depending. Being given one frequency ω (with some restriction for the internal problem) we approach the solution E to the harmonic Maxwell equations by a solution $E_\alpha(t,x)$ to the evolution (time depending) Maxwell system. The control setting is interesting concerning the adjoint problem P which should be a *ingoing* solution (with respect to the Silver-Muller radiating condition at infinity). This difficulty is captured in the calculus of the *Riccati* operator $Q(t)$ so that in the approximation the synthesis of the optimal control eliminates the adjoint states $P(t)$ so that we realise a *physical time inversion* without any such difficulty.

Key-words: Maxwell equations, harmonic regime, Riccati equation

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Une approximation par technique d'optimisation pour les solutions harmoniques en électromagnétisme.

Résumé : Nous considérons un nouveau problème de type optimisation et contrôle optimale pour le calcul des regimes harmoniques en électromagnétisme 3D. Cette approche est robuste dans le sens que les coefficients du milieu peuvent être variable et même varier lentement avec le temps. Etant donnée une fréquence ω on approche la solution E del'equation harmonique par la solution $E_\alpha(t,x)$ du systême d'évolution associé. La structure du contrôle est interessante au sujet de l'état adjoint. Celui-ci doit être olution du problème "entrant" associé à la condition de rayonnement de Silver Muller rentrante. Cette difficulté est localisée dans le calcul de la solution de l'équation de Riccati qui elle ne se fait pas en temps réel. Ainsi la synthese du contrôle optimal éliminant l'état adjoint permet de réaliser ce retournement du temps sans difficulté algorithmique.

Mots-clés : Équations de Maxwell, regime harmonique, équation de Riccati

The caracterisation and calculus of harmonic solution to Maxwell equations in unbounded domain are classiquely done by the use of the stratton-Chu representation formulae. We develop a new control approach for the harmonic solution using the evolution maxwell equation as a "state equation". We show that the time depending solution under appropriate boundary condition on the metallic obstacle B results in 3 usual scalar wave equations, solving together the radiating condition (4.1). For shortness we limitated here the study to metallic obstacle body B (with boundary Γ) but this paper obviously extends to much more general settings including inverse problems in harmonic regime of Maxwell equations.

1 Maxwell equation in bounded domain Ω

Assuming perfect media ($\epsilon = \mu = 1$) the Ampère law is

$$Curl\mathbf{H} = \frac{\partial}{\partial t}E + J, \tag{1.1}$$

where J is the electric current density. The Faraday's law is

$$CurlE = -\frac{\partial}{\partial t}\mathbf{H}. \tag{1.2}$$

The conservations laws are

$$\text{div } E = \rho, \quad \text{div } \mathbf{H} = 0, \tag{1.3}$$

where ρ is the volume charge density. From (1.1) and (1.2), as div $Curl = 0$, we obtain

$$\text{div } J = -\text{div } (E_t) = -\rho_t. \tag{1.4}$$

We assume that $\rho = 0$, which implies that div $J = 0$. Under this assumption any E solving (1.1) is divergence free as soon as the initial condition E_0 is. We shall also assume div $E_0 = 0$ so that (1.6) will be a consequence of (1.1).

With $F = -J_t$, we similarly get div $F = 0$ and E is solution of the usual Maxwell equation:

$$E_{tt} + CurlCurl\, E = F, \quad E(0) = E_0, \quad E_t(0) = E_1. \tag{1.5}$$

Lemma 1.1. *Assume that* div $F =$ div $E_0 =$ div $E_1 = 0$. *Then any solution E to Maxwell equation (1.5) verifies the conservation condition (1.3) (with $\rho = 0$):*

$$\text{div } E = 0. \tag{1.6}$$

We have the classical identity

$$CurlCurl\, E = -\Delta E + \nabla(\text{div } E) \tag{1.7}$$

so that E is also solution of the following wave equation problem

$$E_{tt} - \Delta E = F, \quad E(0) = E_0, \quad E_t(0) = E_1. \tag{1.8}$$

1.1 Boundary Conditions on Γ

The physical boundary condition for metallic boundary is $E \times n = 0$ which can be written as the homogeneous Dirichlet condition on the tangential component of the field E:

$$E_\Gamma = \vec{I} \text{ on } \Gamma. \tag{1.9}$$

We introduce the following Fourier-like boundary condition involving the mean curvature $\Delta b_\Omega = \lambda_1 + \lambda_2$ of the surface Γ

$$\Delta b_\Omega \, E.n + \langle DE.n, n\rangle = -div_\Gamma \vec{I} \text{ on } \Gamma. \tag{1.10}$$

In flat pieces of the boundary this condition reduces to the usual Neumann condition.

Proposition 1.2. *Let E be a smooth element ($E \in \mathcal{H}^2$, see below) and the three divergence-free elements $(E_0,\, E_1,\, F) \in H^2(\Omega, R^3) \times H^1(\Omega, R^3) \times L^2(0,\tau; H^1(\Omega, R^3))$. Then we have the following conclusions.*

i) *Let E be solution to Maxwell-metallic system (1.5), (1.9). Then E solves the mixed wave problem (1.8), (1.9), (1.10) and, from Lemma 1.1, E solves also the free divergence condition (1.6).*

ii) *Let E be solution to the wave equation (1.8) with "metallic" b.c. (1.9). Then E verifies the Fourier-like condition (1.10) if and only if E verifies the free divergence condition (1.6).*

iii) *Let E be a divergence-free solution to the "metallic" wave problem (1.8), (1.6), (1.9), then E solves the Maxwell problem (1.5), (1.9), (1.10).*

Proof:

We consider $e = \text{div } E$; if E is solution to Maxwell problem (1.5) then e solves the scalar wave equation with inital conditions $e_i = \text{div } E_i = 0,\ i = 0, 1$ and right hand side $f = \text{div } F = 0$. If E solves (1.10) then we get $e = 0$, as from the following result we get $e = 0$ on the boundary:

Lemma 1.3. *Let $E \in H^2(\Omega)$ solving the tangential Dirichlet condition (1.9), then we have the following expression for the trace of* div E*:*

$$\text{div } E|_\Gamma = \Delta b_\Omega \,\langle E, n\rangle + \langle DE.n, n\rangle + div_\Gamma \vec{I} \text{ on } \Gamma. \tag{1.11}$$

Proof: The divergence successively decomposes as follows at the boundary :

$$\begin{aligned} \text{div } E|_\Gamma &= \text{div }_\Gamma(E) + \langle DE.n, n\rangle = \text{div }_\Gamma(E.n\,\vec{n}) + \text{div }_\Gamma(E_\Gamma) + \langle DE.n, n\rangle \\ &= \langle \nabla_\Gamma(E.n), n\rangle + E.n \,\text{div }_\Gamma(\vec{n}) + \text{div }_\Gamma(E_\Gamma) + \langle DE.n, n\rangle\,. \end{aligned} \tag{1.12}$$

Obviously $\langle \nabla_\Gamma(E.n), n\rangle = 0$, the mean curvature of the surface Γ is $\Delta b_\Omega = \text{div }_\Gamma(\vec{n})$ and if the field E satisfies the tangential Dirichlet condition (1.9) we get the following simple expression for the restriction to the boundary of the divergence:

$$\text{div } (E)|_\Gamma = \Delta b_\Omega \,\langle E, n\rangle + \langle DE.n, n\rangle + div_\Gamma \vec{I}. \tag{1.13}$$

Then if E satisfies the extra "Fourier-like" condition (1.10) we get $e = 0$ on Γ, so that $e = 0$.

1.2 The Wave-Maxwell Mixed Problem

From previous considerations, it follows that under the divergence-free assumption for the three data E_0, E_1, F, the following three problems are equivalent (in the sense that any smooth solution of one of them is solution to the two others): Maxwell problem (1.5), (1.9), Free-Wave problem (1.8), (1.6), (1.9), and Mixed-Wave problem (1.8), (1.9), (1.10). *We emphasize* that any solution to Maxwell problem satisfies the divergence-free condition (1.6) and the Fourier-like condition (1.10). Any solution to the Mixed-Wave problem satisfies (for free) the divergence-free condition (1.6). Any solution to the Free-Wave problem satisfies (for free) the Fourier-like condition (1.10).

1.2.1 Local distance function

As the boundary $\Gamma = \partial\Omega \in C^2$ we can apply all intrinsic geometry material introduced in ([5]). Denoting by $p = p_\Gamma$ the projection mapping onto the manifold Γ (which is smoothly defined in a tubular neighborhood of Γ) we consider the oriented distance function $b = b_\Omega = d_{\Omega^c} - d_\Omega$ where $\Omega^c = R^N \setminus \Omega$, and its "localized version" defined as follows ([6]): let $h > 0$ be "a small" positive number and $\rho_h(.) \geq 0$ be a cutting scalar smooth function such that $\rho_h(z) = 0$ when $|z| > h$ and $\rho(z) = 1$ when $|z| < h/2$. Then set

$$b_\Omega^h = \rho_h \circ b_\Omega \tag{1.14}$$

and define the associate localized projection mapping

$$p_h = I_d - b_\Omega^h \nabla b_\Omega^h \tag{1.15}$$

smoothly defined in the tubular neighborhood

$$\mathcal{U}_h(\Gamma) = \{x \in D \,:\, |b_\Omega(x)| < h\}. \tag{1.16}$$

1.3 The tangential Field E^τ is solution to Dirichlet problem on Γ

In a first step consider the "tangential vector field" obtained as
$E^\tau = E - E.\nabla b_\Omega^h \, \nabla b_\Omega^h$. We get

$$E^\tau_{tt} - \Delta E^\tau = (E_{tt} - \Delta E) - (E_{tt} - \Delta E).\nabla b_\Omega^h \, \nabla b_\Omega^h + C. \tag{1.17}$$

That is $A.E^\tau = (A.E)^\tau + C$, where the commutator $C \in L^2(0, T, L^2(\Omega, R^3))$ is given by

$$C = -E.\Delta b_\Omega^h \, \nabla b_\Omega^h - 2D^2 b_\Omega^h.\nabla b_\Omega^h \, \nabla b_\Omega^h - E.\nabla b_\Omega^h D^2 b_\Omega^h - 2D^2 b_\Omega^h.\nabla(E.b_\Omega^h). \tag{1.18}$$

The conclusion formally derives as follows: as $E^\tau \in L^2(I, H^1(\Omega, R^3))$ we get the traces terms

$$E^\tau.n = E^\tau_t = 0 \in L^2(I, H^{1/2}(\Gamma)). \tag{1.19}$$

1.4 The Normal Vector Field e is solution to Neumann problem on Γ

Set

$$e = E.\nabla b_\Omega^h. \tag{1.20}$$

Lemma 1.4.

$$e_{tt} - \Delta e = (E_{tt} - \Delta E).\nabla b_\Omega^h + \theta, \tag{1.21}$$

where

$$\theta = D^2 b_\Omega^h..DE + \text{div}\,(D^2 b_\Omega^h.E) \quad \textit{and} \quad \frac{\partial}{\partial n} e = \langle DE.n, n\rangle = -\Delta b_\Omega\, e \;\; \textit{on}\; \Gamma. \tag{1.22}$$

Then e is solution of the wave problem:

$$e_{tt} - \Delta e = \Theta, \tag{1.23}$$

where

$$\Theta = F.\nabla b_\Omega^h + D^2 b_\Omega^h..DE + \text{div}\,(D^2 b_\Omega^h.E). \tag{1.24}$$

1.5 Harmonic regime

Under periodical tangential current g on Γ, say $g(t,x) = exp(-i\omega t)\, G(x)$ and $F = 0$, we look for solution in the form $\bar{\mathcal{E}}(t,x) = exp(-i\omega t)\, E(x)$ in the bounded domain Ω to the wave problem 1.5, 1.9, 1.10. Necessarely the initial conditions (E_0, E_1) are in the following form

$$(E_0, E_1) = (Y, -i\omega Y), \;\; Y \in \mathcal{K}_G,$$

Where

$$\mathcal{K}_G = \{\, Y \in H^2(\Omega, C^3), \quad with$$
$$H\, Y.n + < DY.n, n >= -div_\Gamma G, \quad Y \times n = G \; on \; \Gamma \,\}$$

2 Non zero mean curvature

Assume now that the domain Ω has a $C^{1,1}$ boundary Γ such that there exists $\alpha > 0, \beta > 0$ with

$$a.e.x \in \Gamma, \;\; \alpha \le |\Delta b_\Omega(x)| \; \le \beta$$

Then the previous boudary condition derives to the following "implicit Dirichlet Boundary Condition"

$$Y_\Gamma = \vec{g}$$
$$Y.n = -\frac{1}{H}\, h$$

Where

$$h =< DY.n, n >$$

That is

$$Y = \vec{g} + h\,\vec{n} \;\; on \;\; \Gamma$$

2.1 Wave with Dirichlet condition $\vec{g}+\, h\,\vec{n}$ on the bounded domain boundary

We denote by $H^1_\Gamma(\Omega)$ the space of all $f \in H^1(\Omega)$ such that $f = 0$ on Γ. Thanks to the Poincaré inequality, a norm om $H^1_\Gamma(\Omega)$, equivalent to the $H^1(\Omega)$-norm, is given by

$$\|\phi\| = \sqrt{\int_\Omega |\nabla\phi|^2\, dx}, \quad \phi \in H^1_\Gamma(\Omega).$$

Fix $g \in H^{1/2}(\Gamma)$ and $y \in H^1(\Omega)$ such that $y|_\Gamma = g$. We consider the wave problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\mathcal{E} - \Delta\mathcal{E} = Fe^{-i\omega t}, \quad \text{in } [0,T]\times\Omega, \\ \mathcal{E}(t)|_\Gamma = ge^{-i\omega t} \in H^{1/2}(\Gamma),\, t \in [0,T] \\ \mathcal{E}(0) = y \\ \mathcal{E}_t(0) = -i\omega y. \end{cases} \tag{2.1}$$

where $F \in (H^1_\Gamma(\Omega))^*$ Notice that if $y \in H^1(\Omega)$ and $\exists \Delta y \in L^2(\Omega)$, then $\partial y/\partial n$ is well defined on $\partial\Omega$ and it belongs to $H^{1/2}(\partial\Omega)$.

For a fixed $g \in H^{1/2}(\Gamma)$, our problem is find the unique $y^* \in H^1(\Omega)$ such that $y^*|_\Gamma = g$ and such that the solution of (2.1) has the harmonic regime, that is $\mathcal{E}(t) = \Re e(y^* e^{-i\omega t})$. It is well known that such a condition is equivalent so solve a Helmholz type equation. We want to find it through an optimization method.

2.1.1 Choice of ω^2

It is well known and easy to show that the harmonic state y solves the Helmholz equation

$$\begin{cases} \omega^2 y + \Delta y = F \quad \text{on } \Omega \\ y = g \quad \text{on } \Gamma. \end{cases}$$

By Fredholm alternative, we have to distinguish two cases: when ω^2 is a eigenvalue of the laplacian with zero dirichlet boundary condition or when it is not.

In the first case, if $g \neq 0$, the problem is ill-posed. Indeed, if z is an eigenvector correspondent to ω^2, then y solves the Helmholz equation if and only if $Y = y - z$ solves

$$\begin{cases} \omega^2 Y + \Delta Y = -\omega^2 z \quad \text{on } \Omega \\ Y = 0 \quad \text{on } \Gamma. \end{cases}$$

which is clearly not solvable.

Then we have to assume that ω^2 is not a eigenvalue. If this is the case, then the Helmholz equation has a unique solution of the form $y = X + Y$, where X solves the Helmholz equation with zero Dirichlet boundary conditions and Y solves the homogeneous problem with $Y = g$ on Γ.

2.2 Minimization

For any time interval $[0, \tau]$, $\tau > 0$ and $\Omega' \subset \Omega$ we consider here the functional

$$J(h, Y) = \frac{1}{2}\, \Sigma_{k=1,3}(\int_0^T \int_{\Omega'} \left|\mathcal{E}_k(t; Y_k)(x) - Y_k(x) e^{-i\omega t}\right|^2 dx dt)$$

$$+ \int_\Gamma |< DY.n, n > -h\,|^2\, d\Gamma$$

Where $\mathcal{E}_k(t; Y_k)$ is the solution of (2.1) with $g = (\vec{I} + h\, n)_k$, $k = 1, 3$ and initial condition Y_k. We look at Y^*, h^* as the unique solution of $J(Y^*, h^*) = 0$, then, we consider the optimization problem

$$\inf_{h \in H^{1/2}(\Gamma),\ Y \in H^1(\Omega): Y|_\Gamma = \vec{g} + h\, \vec{n}} J(h, Y).$$

Obviously this problem can be decomposed as follows:

$$Inf\{\ j(h),\ \ h \in H^{1/2}(\Gamma)\ \ \},$$

where

$$j(h) = Inf\{\ J(h, Y),\ \ Y \in H^1(\Omega),\, \Delta Y \in L^2(\Omega, R^N),\ < DY.n, n >= h\ \ \}$$

3 Unbounded domain

4 Radiation condition in unbounded domain

We consider a bounded rigid body $B \subset R^3$ with smooth boundary Γ E, H denotes the electromagnetic fields in the unbounded domain $B^c = R^3 \setminus \bar{B}$. We assume that $0 \in B$ and denotes $r = |x|_{R^3}$. The *radiating* electromagnetic field solves the equations (1.1), (1.1), (1.5) bellow and the Silver-Müler conditions when $r \to \infty$:

$$E \text{ and } H = O(\frac{1}{r}),\ \ E + x \times H \text{ and } H - x \times E\ = o(\frac{1}{r}) \tag{4.1}$$

For a mathematical treatment of the electromagnetic problem we refer to Nédélec [2] who proposed variational formulations with nonlocal boundary conditions on an artificial boundary S based on the representation by the exterior capacity operator $T_S \in \mathcal{L}(H^{1/2}(S, R^3), H^{-1/2}(S, R^3))$.

This operator is explicitly known when S is a sphere, say $S = \{x \in R^3,\ |x| = R\}$. The unbounded domain B^c will then be "replaced" by the bounded domain $\Omega = B^c \cap \{x \in R^3,\ |x| < R\}$ with boundary $\Gamma \cup S$, with a new boundary condition of Neumann kind on S replacing (4.1) and involving T_S. In the following analysis the boundary S is not necessarily a sphere so we more generally assume that $D \subset \mathbb{R}^N$ is a bounded domain, $B \subset D$ with $\bar{B} \subset D$ and $\Omega = D \setminus \bar{B}$. The boundary $\partial\Omega$ of Ω is splitted into the boundary Γ of B and the boundary S of D: $\partial\Omega = S \cup \Gamma,\ S \cap \Gamma = \emptyset$. We also assume that ∂B and Γ are C^2 regular.

Assuming perfect media ($\epsilon = \mu = 1$) the Ampère law is

$$Curl\mathbf{H} = \frac{\partial}{\partial t}E + J, \tag{4.2}$$

where J is the electric current density. The Faraday's law is

$$CurlE = -\frac{\partial}{\partial t}\mathbf{H}. \tag{4.3}$$

The conservations laws are

$$\text{div } E = \rho, \quad \text{div } \mathbf{H} = 0, \tag{4.4}$$

where ρ is the volume charge density. From (4.2) and (4.3), as div $Curl = 0$, we obtain

$$\text{div } J = -\text{div } (E_t) = -\rho_t. \tag{4.5}$$

We assume that $\rho = 0$, which implies that div $J = 0$. Under this assumption any E solving (4.2) is divergence free as soon as the initial condition E_0 is. We shall also assume div $E_0 = 0$ so that (1.6) will be a consequence of (4.2).

With $F = -J_t$, we similarly get div $F = 0$ and E is solution of the usual Maxwell equation:

$$E_{tt} + CurlCurl\, E = F, \quad E(0) = E_0, \quad E_t(0) = E_1. \tag{4.6}$$

Lemma 4.1. *Assume that* div $F =$ div $E_0 =$ div $E_1 = 0$*. Then any solution E to Maxwell equation (4.6) verifies the conservation condition (4.4) (with $\rho = 0$):*

$$\text{div } E = 0. \tag{4.7}$$

We have the classical identity

$$CurlCurl\, E = -\Delta E + \nabla(\text{div } E) \tag{4.8}$$

so that E is also solution of the following wave equation problem

$$E_{tt} - \Delta E = F, \quad E(0) = E_0, \quad E_t(0) = E_1. \tag{4.9}$$

4.1 Radiating Conditions and condition on Γ

The physical boundary condition for metallic boundary is $E \times n = 0$ which can be written as the homogeneous Dirichlet condition on the tangential component of the field E:

$$E_\Gamma = \vec{g} \text{ on } \Gamma. \tag{4.10}$$

We introduce the following Fourier-like boundary condition involving the mean curvature $\Delta b_\Omega = \lambda_1 + \lambda_2$ of the surface Γ

$$\Delta b_\Omega\, E.n + \langle DE.n, n\rangle = -div_\Gamma \vec{g} \text{ on } \Gamma. \tag{4.11}$$

In flat pieces of the boundary this condition reduces to the usual Neumann condition.

Proposition 4.2. *Let E be a smooth element ($E \in \mathcal{H}^2$, see below) and the three divergence-free elements $(E_0,\, E_1,\, F) \in H^2(\Omega, R^3) \times H^1(\Omega, R^3) \times L^2(0, \tau; H^1(\Omega, R^3))$. Then we have the following conclusions.*

i) *Let E be solution to Maxwell-metallic system (4.6), (4.10), (4.1). Then E solves the mixed wave problem (4.9), (4.10), (4.11), (4.1) and, from Lemma 4.1, E solves also the free divergence condition (4.7).*

ii) *Let E be solution to the wave equation (4.9), (4.1) with "metallic" b.c. (4.10). Then E verifies the Fourier-like condition (4.11) if and only if E verifies the free divergence condition (4.7).*

iii) *Let E be a divergence-free solution to the "metallic" wave problem (4.9), (4.7), (4.1), (4.10), then E solves the Maxwell problem (4.6), (4.10), (4.11), (4.1).*

Proof:

We consider $e = \text{div } E$; if E is solution to Maxwell problem (4.6) then e solves the scalar wave equation with initial conditions $e_i = \text{div } E_i = 0,\ i = 0, 1$ and right hand side $f = \text{div } F = 0$. If E solves (4.11) then we get $e = 0$, as from the following result we get $e = 0$ on the boundary:

Lemma 4.3. *Let $E \in H^2(\Omega)$ solving the tangential Dirichlet condition (4.10), then we have the following expression for the trace of* div E*:*

$$\text{div } E|_\Gamma = \Delta b_\Omega \langle E, n\rangle + \langle DE.n, n\rangle + div_\Gamma \vec{g} \text{ on } \Gamma. \tag{4.12}$$

Proof: The divergence successively decomposes as follows at the boundary :

$$\begin{aligned} \text{div } E|_\Gamma &= \text{div }_\Gamma(E) + \langle DE.n, n\rangle = \text{div }_\Gamma(E.n\,\vec{n}) + \text{div }_\Gamma(E_\Gamma) + \langle DE.n, n\rangle \\ &= \langle \nabla_\Gamma(E.n), n\rangle + E.n \,\text{div }_\Gamma(\vec{n}) + \text{div }_\Gamma(E_\Gamma) + \langle DE.n, n\rangle\,. \end{aligned} \tag{4.13}$$

Obviously $\langle \nabla_\Gamma(E.n), n\rangle = 0$, the mean curvature of the surface Γ is $\Delta b_\Omega = \text{div }_\Gamma(\vec{n})$ and if the field E satisfies the tangential Dirichlet condition (4.10) we get the following simple expression for the restriction to the boundary of the divergence:

$$\text{div } (E)|_\Gamma = \Delta b_\Omega \langle E, n\rangle + \langle DE.n, n\rangle + div_\Gamma \vec{g}. \tag{4.14}$$

Then if E satisfies the extra "Fourier-like" condition (4.11) we get $e = 0$ on Γ, so that $e = 0$.

4.2 The Mixed Problem

From previous considerations, it follows that under the divergence-free assumption for the three data E_0, E_1, F, the following three problems are equivalent (in the sense that any smooth solution of one of them is solution to the two others): Maxwell problem (4.6), (4.10), Free-Wave problem (4.9), (4.7), (4.10), and Mixed-Wave problem (4.9), (4.10), (4.11). *We emphasize* that any solution to Maxwell problem satisfies the divergence-free condition (4.7) and the Fourier-like condition (4.11). Any solution to the Mixed-Wave problem satisfies (for free) the divergence-free condition (4.7). Any solution to the Free-Wave problem satisfies (for free) the Fourier-like condition (4.11).

5 Harmonic regime

Under periodical tangential current g on Γ, say $g(t,x) = exp(-i\omega t)\, G(x)$ and $F = 0$, we look for solution in the form $\bar{\mathcal{E}}(t,x) = exp(-i\omega t)\, E(x)$ in the unbounded domain B^c to the radiating wave problem 4.6, 4.10, 4.11, 4.1. Necessarily the restrictions to the bounded domain Ω of the initial conditions (E_0, E_1) are in the following form

$$(E_0, E_1) = (Y, -i\omega Y),\ \ Y \in \mathcal{K}_G,$$

Where

$$\mathcal{K}_G = \{\, Y \in H^2(\Omega, C^3),\ divY = 0,\ with$$
$$\text{curl }\ Y \times n = T_S.Y\ on\ S,\ and\ Y \times n = G\ on\ S\ \}$$

5.1 Maxwell equations in $\Omega = B^c \cap \{\, |x| < R \,\}$

By the use of the capacity operator T_S we derive a formulation of the harmonic. Then we consider the problem

$$\omega^2 Y + \Delta Y = 0\ in\ \Omega \tag{5.1}$$

$$Y \times n = G\ on\ \Gamma \tag{5.2}$$

$$\Delta b_\Omega\, Y.n + < DY.n, n > = 0\ on\ \Gamma \tag{5.3}$$

$$\tag{5.4}$$

In order to derive the expression of Y from the calculus of the evolution solution $\bar{\mathcal{E}}(t,x)$ we introduce the "radiating wave problem in the bounded domain Ω " as being the vector field

$\mathcal{E}(t,x)$ solution to the system 4.6 in Ω, 4.10 and 4.11 on Γ , and the following Neumann condition on the *artificial boundary* S:

$$\text{curl } \mathcal{E} \times n = T_S.Y \text{ } on \text{ } I \times S \tag{5.5}$$

With initial conditions

$$\mathcal{E}(0) = Y, \frac{\partial}{\partial t}\mathcal{E}(0) = -i\omega Y$$

Then, we consider the optimization problem

$$\inf_{Y \in H^1(\Omega, C^3): Y|_\Gamma = g} J(Y).$$

Where the functional is chosen in the following form:

$$J(Y) = \frac{1}{2}\int_0^\tau \int_\Omega \left|\mathcal{E}(t;Y)(x) - Y(x)e^{-i\omega t}\right|^2 dxdt.$$

And where $Y|_\Gamma$ is the tangential component of the vector Y on Γ.

6 The situation for Scalar Wave

As usual, $L^k(\Omega)$, $1 \le k < \infty$ is the space of all the Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ such that

$$\|f\|_{L^k} = \left(\int_\Omega |f(x)|^k dx\right)^{\frac{1}{k}} < \infty.$$

If $k = \infty$, then $L^\infty(\Omega)$ is the space of all Lebesgue measurable functions such that

$$\|f\|_{L^\infty} = \inf\{\lambda > 0 : mes(\{|f| > \lambda\}) = 0\} < \infty.$$

The norm in $L^k(\Omega)$ is denoted by $\|\cdot\|_{L^k}$. For any $m, k \in \mathbb{N}$, $W^{m,k}(\Omega)$ is the space of all the Lebesgue measurable functions whose all weak derivative, up to the order m, belongs to L^k. The spaces $W^{m,k}(\Omega)$ are endowed by the usual Sobolev norm, which we denote by $\|\cdot\|_{W^{m,k}}$. When $k = 2$, we set $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The space of fractional order are defined via complex interpolation.

We denote by $H^1_\Gamma(\Omega)$ the space of all $f \in H^1(\Omega)$ such that $f = 0$ on Γ. Thanks to the Poincaré inequality, a norm om $H^1_\Gamma(\Omega)$, equivalent to the $H^1(\Omega)$-norm, is given by

$$\|\phi\| = \sqrt{\int_\Omega |\nabla\phi|^2 \, dx}, \quad \phi \in H^1_\Gamma(\Omega).$$

Fix $g \in H^{1/2}(\Gamma)$ and $y \in H^1(\Omega)$ such that $y|_\Gamma = g$. We consider the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\mathcal{E} - \Delta\mathcal{E} = Fe^{-i\omega t}, \quad \text{in } [0,T]\times\Omega, \\ \mathcal{E}(t)|_\Gamma = ge^{-i\omega t} \in H^{1/2}(\Gamma),\ t\in[0,T] \\ \dfrac{\partial\mathcal{E}}{\partial n}(t)|_S = Tye^{-i\omega t} \in H^{-1/2}(S),\ t\in[0,T] \\ \mathcal{E}(0) = y \\ \mathcal{E}_t(0) = -i\omega y. \end{cases} \tag{6.1}$$

where $F \in (H^1_\Gamma(\Omega))^*$ and $T : H^{1/2}(S) \to H^{-1/2}(S)$ is the *capacity operator*.

Notice that if $y \in H^1(\Omega)$ and $\exists\Delta y \in L^2(\Omega)$, then $\partial y/\partial n$ is well defined on $\partial\Omega$ and it belongs to $H^{1/2}(\partial\Omega)$.

For a fixed $g \in H^{1/2}(\Gamma)$, our problem is find the unique $y^* \in H^1(\Omega)$ such that $y^*|_\Gamma = g$ and such that $\mathcal{E}(t) = \Re e(y^* e^{-i\omega t})$. It is well known that such a condition is equivalent so solve a Helmholz type equation. We want to find it through an optimization method.

Our idea consists is looking at y^* as the unique solution of $J(y^*) = 0$, where J is the functional

$$J(y) = \frac{1}{2}\int_0^T \int_\Omega \left|\mathcal{E}(t;y)(x) - y(x)e^{-i\omega t}\right|^2 dxdt.$$

Then, we consider the optimization problem

$$\inf_{y\in H^1(\Omega):y|_\Gamma=g} J(y).$$

More generally for any time interval $[0,\tau]$, $\tau > 0$ and $\Omega' \subset \Omega$ we could consider

$$J_{\tau,\Omega'}(y) = \int_0^\tau \int_{\Omega'} \left|\mathcal{E}(t;y)(x) - y(x)e^{-i\omega t}\right|^2 dxdt,$$

which reaches zero at the harmonic solution.

7 Regularity for Wave mixed Dirichlet-Neumann Boundary Problem solution $\mathcal{E}$

We obtain regularity for the solution $\mathcal{E}$ of the wave equation with the Neumann condition $\frac{\partial\mathcal{E}}{\partial n} = h$ on the boundary S. Let Ω be a bounded open domain in R^N with Lipschiztian boundary :

$$\partial\Omega = \Gamma \cup S,\ \ \bar{S}\cap\bar{\Gamma} = \emptyset,$$

and $A(x) = \{a_{ij}(x)\}$ be a matrix of functions defined on Ω with the following properties:

(i) $a_{ij} \in L^\infty(\Omega)$

(ii) $\exists\alpha > 0$ such that $\forall x \in \Omega$, $\forall\zeta = \{\zeta_i\} \in R^N$, $a_{ij}\,\zeta_i\zeta_j \geq \alpha|\zeta|^2$.

Associate with the matrix $A(x)$ the operator

$$A\mathcal{E} = div(A(x)\nabla\mathcal{E}).$$

Given $\tau > 0$ and the interval $I = [0,\,\tau]$, consider the following wave equation problem

$$\mathcal{E}_{tt} + A.\mathcal{E} = f, \quad \frac{\partial\mathcal{E}}{\partial n_A} = h, \ on\ S, \ \mathcal{E} = g\ on\ \Gamma, \quad \mathcal{E}(0) = \mathcal{E}_0, \quad \mathcal{E}_t(0) = \mathcal{E}_1. \tag{7.1}$$

7.1 Linearity and partial regularity

We decompose, as usual for linear problem, the regularity in two steps: $\mathcal{E} = \mathcal{E}^0 + Z$ where Z is the solution with $h = 0$ while $\mathcal{E}^0$ is the solution for $f = \mathcal{E}_0 = \mathcal{E}_1 = g = 0$. For the first element, solution to the homogeneous Neumann boundary condition on S, the literature furnishes all the hoped regularity: the semi group theory and the sharp boundary regularity of Lasiecka and Triggiani hold. Then all the difficulty is concentrated in the second element. This means that we can directly drop to zero the non essential data , making $f = \mathcal{E}_0 = \mathcal{E}_1 = 0$. We shall assume in the sequel that the Neumann data on S verifies $h \in W^{1,1}(I, H^{-1/2}(S))$ which is verified in our context as we has $h(t,x) = exp(-i\omega t)g(x)$ and g is given in $H^{-1/2}(S)$.

7.1.1 Galerkin solution $\mathcal{E}^0 \in L^2(I, H^1_\Gamma(\Omega)) \cap H^1(I, L^2(\Omega))$

Proposition 7.1. *Let $F = g = \mathcal{E}_0 = \mathcal{E}_1 = 0$ and $h \in W^{1,1}(I, H^{-1/2}(S))$. There exists a solution $\mathcal{E} \in L^2(I, H^1_\Gamma(\Omega)) \cap H^1(I, L^2(\Omega))$ to the wave-Neumann problem (7.1).*

Proof: Let us consider any dense numerable family $e_1, ..., e_m, ...$ and consider the element $\mathcal{E}^m = \Sigma_{i=1,\dots,m}\ \mathcal{E}^m_i \otimes e_i$, that is :

$$\mathcal{E}^m(t,x) = \Sigma_{i=1,\dots,m}\ \mathcal{E}^m_i(t)\, e_i(x)$$

solution to the ordinary second order linear differential system:

$$\int_\Omega \left((\frac{\partial^2}{\partial t^2}\mathcal{E}^m)e_j + < A.\nabla\mathcal{E}^m, \nabla e_j > \right) dx = \int_S h(t,x)\, e_j(x)\, dS(x),$$

$$\forall t \in I,\ \forall j,\ 1 \leq j \leq m.$$

With adapted initial conditions :

$$\mathcal{E}^m(0) = 0,\ \frac{\partial}{\partial t}\mathcal{E}^m(0) = 0$$

We classiquely consider the energy term $E^m(t) = 1/2\ \int_\Omega((\frac{\partial}{\partial t}\mathcal{E}^m)^2 + |\nabla\mathcal{E}^m|^2\,)dx$, and we get

$$\frac{\partial}{\partial t}E^m(t) = \int_S h(t,x)\,\frac{\partial}{\partial t}\mathcal{E}^m(t,x)\,dS(x).$$

$$E^m(t) = -\int_S \int_0^t \frac{\partial}{\partial t} h(s,x)\, \mathcal{E}^m(s,x) ds\, dS(x)$$
$$+\int_S h(t,x)\, \mathcal{E}^m(t,x)\, dS(x) - \int_S h(0,x)\, \mathcal{E}^m(0,x)\, dS(x).$$

So that assuming $h(0,.) = 0$, we get:

$$E^m(t) \leq \|h(t)\|_{H^{-1/2}(S)}\, \|\mathcal{E}^m(t)\|_{H^{1/2}(S)} + \int_0^t \|\frac{\partial}{\partial t} h(t)\|_{H^{-1/2}(S)}\, \|\mathcal{E}^m(t)\|_{H^{1/2}(S)} dt$$
$$\leq c\, \|h(t)\|_{H^{-1/2}(S)}\, \|\mathcal{E}^m(t)\|_{H^1_\Gamma(\Omega)} + c\int_0^t \|\frac{\partial}{\partial t} h(t)\|_{H^{-1/2}(S)}\, \|\mathcal{E}^m(t)\|_{H^1_\Gamma(\Omega)} dt,$$
$$\leq c\, \|h(t)\|_{H^{-1/2}(S)}\, \|\mathcal{E}^m(t)\|_{H^1_\Gamma(\Omega)} + c\int_0^T \|\frac{\partial}{\partial t} h(t)\|_{H^{-1/2}(S)} dt\, \|\mathcal{E}^m(t)\|_{L^\infty(I,H^1_\Gamma(\Omega))},$$

from which we derive that

$$E^m(t) \leq c\, \|h(t)\|_{H^{-1/2}(S)}\, E^m(t)^{1/2} + c \sup_{0<t<T}\; E^m(t)^{1/2}\, \|\frac{\partial}{\partial t} h(t)\|_{L^1(I,H^{-1/2}(S))}.$$

Then

$$\|E^m\|_{L^\infty(I)} \leq\; c\, (\|h(t)\|_{L^\infty(I,H^{-1/2}(S))} \;+\; \|\frac{\partial}{\partial t} h(t)\|_{L^1(I,H^{-1/2}(S))}\,)\, \|E^m\|^{1/2}_{L^\infty(I)},$$

from which we derive that the energy norm $\|E^m\|_{L^\infty(I)}$ remains bounded as $m \to \infty$. Then the sequences $\frac{\partial}{\partial t}\mathcal{E}^m$, $\nabla\mathcal{E}^m$ remains bounded in $L^2(I \times \Omega)$ and the weak limiting element is classiquely solution to the problem (7.1).

7.2 The capacity operator T

Denote by Ω_e the exterior domain $\Omega_e = \mathbb{R}^3 \setminus \bar{D}$. The capacity operator $T : H^{1/2}(S) \to H^{-1/2}(S)$ is defined by

$$Tu = \frac{\partial y}{\partial n},$$

where y is solution of the Helmholtz equation on $\mathbb{R}^3 \setminus D$ with Dirichlet conditions on S equal to u and Sommerfeld radiation conditions. That is, y is solution of the problem

$$\begin{cases} k^2 y + \Delta y = 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\ y = u, \quad \text{on } S, \\ \lim\limits_{r\to\infty} \displaystyle\int_{\partial B_r} \left|\frac{\partial y}{\partial r} - iky\right|^2 ds(x) = 0. \end{cases} \tag{7.2}$$

and n the unit normal vector on the boundary $\partial\Omega$, directed into the unbounded domain $\mathbb{R}^3 \setminus D$.

Here and in the following, n the unit normal vector on the boundary $\partial\Omega$ and directed into the unbounded domain $\mathbb{R}^3 \setminus D$.

According to [2, Theorem 2.6.6], there exists a unique variational solution $y \in H^1(\Omega)$ of problem (7.2) which satisfies

$$\int_{\mathbb{R}^3\setminus D} (\nabla y \cdot \nabla v)dx - k^2 \int_{\mathbb{R}^3\setminus D} yvdx = \int_S Tyvd\sigma \quad dx, \quad \forall v \in \{H^1(\mathbb{R}^3 \setminus D); v_{|S} = 0\}.$$

The operator T is linear and continuous, and it satisfies the coercitivity and continuity inequalities

$$\begin{aligned} \|u\|_{L^2(S)} &\leq -\Re e(Tu,u)_{L^2(S)} \leq \|u\|_{H^{1/2}(S)} \\ 0 &\leq \Im m(Tu,u)_{L^2(S)} \leq k\|u\|_{L^2(S)}, \end{aligned}$$

see [2, Chapter 2, §6.3]. We now give a representation of T^*.

Theorem 7.2. *For any $v \in H^{1/2}(S)$ it holds*

$$T^*v = \frac{\partial z}{\partial n},$$

where z is solution of the exterior problem with conjugated Sommerfeld radiating conditions

$$\begin{cases} k^2 z + \Delta z = 0 \quad in\ \mathbb{R}^3 \setminus D, \\ z = v, \quad on\ S, \\ \lim\limits_{r\to\infty} \displaystyle\int_{\partial B_r} \left|\frac{\partial z}{\partial r} + ikz\right|^2 ds(x) = 0. \end{cases} \tag{7.3}$$

Moreover, $\overline{T^*} = T$.

Proof. Take $u, v \in H^{1/2}(S)$ and let $y \in H^1(\Omega_e)$ be solution of the exterior problem (7.2) with Dirichlet boundary condition on S equal to u, and let $z \in H^1(\Omega_e)$ be the solution of (7.3) with Dirichlet boundary conditions equal to v. Fix r big enough to have $S \subset B_r$. By Gauss formula we have

$$\int_{\partial(\Omega_e\cap B_r)} \left(\bar z \frac{\partial y}{\partial n} - y\frac{\partial \bar z}{\partial n}\right) ds(x) =$$

$$= \int_S \left(\bar z Ty - y\frac{\partial \bar z}{\partial n}\right) ds(x) + \int_{\partial B_r} \left(\bar z \frac{\partial y}{\partial n} - y\frac{\partial \bar z}{\partial n}\right) ds(x)$$

$$= \int_{\Omega_e\cap B_r} (\bar z\Delta y - y\Delta \bar z)\, dx = \int_{\Omega_e\cap B_r} \big(\bar z(k^2 + \Delta y) - y(k^2 + \Delta \bar z)\big)\, dx = 0$$

since y, z solves (7.2), (7.3), respectively. Then,

$$\int_S \left(\bar z Ty - y\frac{\partial \bar z}{\partial n}\right) ds(x) = \int_{\partial B_r} \left(y\frac{\partial \bar z}{\partial n} - \bar z\frac{\partial y}{\partial n}\right) ds(x).$$

Moreover,

$$\int_{\partial B_r}\left(y\frac{\partial \bar z}{\partial n}-\bar z\frac{\partial y}{\partial n}\right)ds(x)=$$

$$=\int_{\partial B_r}\frac{y}{r}\left(\overline{\frac{\partial z}{\partial r}+ikz}\right)ds(x)-\int_{\partial B_r}\frac{\bar z}{r}\left(\frac{\partial y}{\partial r}-iky\right)ds(x) \tag{7.4}$$

since $\partial y/\partial r = r\nabla y\cdot n$.

By taking into account that

$$\frac{z(x)}{x},\ \frac{y(x)}{x}\in L^2(\Omega_e)$$

see [2], pag 102, and that y, z are harmonic (hence, continuous) in Ω_e, it follows that

$$\lim_{r\to\infty}\int_{\partial B_r}\frac{|z(x)|^2}{r^2}ds(x)=\lim_{r\to\infty}\int_{\partial B_r}\frac{|y(x)|^2}{r^2}ds(x)=0.$$

This, together to the fact that y, z satisfies the radiating conditions (7.2), (7.3) imply that the last term of (7.4) vanishes as $r\to\infty$. Therefore,

$$\langle Tu, v\rangle_{\sigma(H^{-1/2},H^{1/2})}\int_S \bar v Tu ds(x)=\int_S u\frac{\partial \bar z}{\partial n}ds(x),$$

for any $u, v\in H^{1/2}(S)$. This implies that

$$T^*v=\frac{\partial z}{\partial n},$$

where z is solution of (7.3).

Finally, notice that that if y solves (7.2), then $\bar y$ solves (7.3) with the boundary condition $\bar y=\bar u$. Hence,

$$Tu=\frac{\partial y}{\partial n}=\overline{\frac{\partial \bar y}{\partial n}}=\overline{T^*\bar u}=\overline{T^*}u. \qquad \square$$

7.3 The space K_0

For any $g\in H^{1/2}(\Omega)$, the set K_g consists in all $u\in H^1(\Omega)$ such that $\Delta u\in L^2(\Omega)$ and $u=g$ on Γ, $\partial u/\partial n = Tu$ on S.

Proposition 7.3. *The set K_0 is dense in $H^1_\Gamma(\Omega)$.*

Proof. We have to show that if $f\in H^1_\Gamma(\Omega)$ and

$$\int_\Omega \nabla u\cdot\nabla f=0,\quad \forall u\in K_0$$

then $f=0$. Notice that if u is $C^\infty(\Omega)$ with compact support in Ω, then we find

$$\int_\Omega u\Delta f = 0.$$

This implies $\Delta f = 0$ in the sense of distribution on Ω, then $\Delta f = 0$ as function in $L^2(\Omega)$. Since $f \in H^1(\Omega)$ and $\Delta f \in L^2(\Omega)$, it follows that $H^{-1/2}(S)$. Still by Green's formula,

$$\int_\Omega \nabla u \cdot \nabla f = \int_S u \frac{\partial f}{\partial n} = 0,$$

for any $u \in K_0$. But

$$\{u|_S,\, u \in K_0\} = H^{1/2}(S),$$

therefore $\partial f/\partial n = 0$. We have found that f is solution of

$$\begin{cases} \Delta f = 0, & \text{in } \Omega, \\ f = 0, & \text{on } \Gamma, \\ \dfrac{\partial f}{\partial n} = 0 & \text{on } S. \end{cases}$$

This clearly implies $f = 0$. □

7.4 Optimality System

We set

$$E(t,x) = \mathcal{E}(t;y)(x) - y(x)e^{-i\omega t} - z. \tag{7.5}$$

where $z = z(t)$ is solution of

$$\begin{cases} z_{tt} - \Delta z = F e^{-i\omega t}, \\ z|_\Gamma = 0,\ t \in [0,T] \\ \dfrac{\partial z}{\partial n}(t)|_S = 0,\ t \in [0,T] \\ z(0) = z_t(0) = 0. \end{cases}$$

If $F \in H^1(\Omega)$, standard results show that $z \in C([0,T];H^2) \cap C^1([0,T];H^1)$. We also introduce the operator

$$B(t)y = e^{-i\omega t}(w^2 + \Delta)y.$$

Notice that

$$\int_\Omega By(t)\overline{f}dx = \int y\overline{e^{i\omega t}(w^2+\Delta)f}dx + \int_{\partial\Omega} e^{-i\omega t}\left(\overline{f}\frac{\partial y}{\partial n} - y\overline{\frac{\partial f}{\partial n}}\right) ds(x). \tag{7.6}$$

The function E is solution of

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} E - \Delta E = B(t)y, \quad \text{in } [0,T] \times \Omega, \\ E(t)|_\Gamma = 0,\ t \in [0,T] \\ \dfrac{\partial E}{\partial n}(t)|_S = \left(Ty - \dfrac{\partial y}{\partial n} \right) e^{-i\omega t},\ t \in [0,T] \\ E(0) = 0 \\ E_t(0) = 0. \end{cases} \tag{7.7}$$

The functional becomes

$$J_\alpha(y) = \frac{1}{2} \int_0^T \int_\Omega |E(t;y)(x) + z(t;x)|^2 \, dxdt + \frac{\alpha}{2} \int_\Omega (|y(x)|^2 + |\nabla y(x)|^2) dx.$$

It is useful to introduce the linearization of problem (7.7). We denote by $\eta(t)R = D_y E(t;y) \cdot R$ the Gateaux differential of $E(t;y)$ along the direction $R \in K_0$. Standard results show that $\eta(t)R$ is solution of equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} \eta(t)R - \Delta \eta(t)R = B(t)R, & \text{in } [0,T] \times \Omega, \\ \eta(t)R|_\Gamma = 0, & t \in [0,T] \\ \dfrac{\partial \eta(t)R}{\partial n}|_S = \left(TR - \dfrac{\partial R}{\partial n} \right) e^{-i\omega t}, & t \in [0,T] \\ \eta(0)R = 0 \\ \eta_t(0)R = 0. \end{cases}$$

The solution $\eta(t)R$ corresponds, formally, to the Gateaux derivative of E with respect to y along the direction R. Notice that $\eta(t)R$ does not depends by y. Finally, we introduce the adjoint equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} P(t;y) - \Delta P(t;y) = E(t;y) + z(t), & \text{in } [0,T] \times \Omega, \\ P(t;y)|_\Gamma = 0, & t \in [0,T] \\ \dfrac{\partial P(t;y)}{\partial n}|_S = 0, & t \in [0,T] \\ P(T;y) = 0 \\ P(T;y) = 0. \end{cases}$$

Notice that $P(t;y)$ it is not linear in y.

Let us back to the functional J_α. Since E is linear, the functional J_α is strictly convex in $H^1(\Omega)$. Then there exists a unique minimizing element. That is, we have the following

Theorem 7.4. *For any $\alpha > 0$, there exists a unique $y^*_\alpha \in H^1(\Omega)$, $y^*_\alpha|_\Gamma = g$ which minimizes J_α, i.e.*

$$J_\alpha(y^*_\alpha) = \min_{y \in H^1(\Omega), y|_\Gamma = g} J_\alpha(\Omega).$$

We now look for a representation of y_α^*. To do this, we need to introduce the operator A_α, for $\alpha > 0$. The operator A_α is defined as follows: for $x \in C([0,T]; H^2(\Omega))$, $A_\alpha x$ is the variational solution of the problem

$$\begin{cases} (I-\Delta)w = -\dfrac{1}{\alpha}\displaystyle\int_0^T e^{i\omega t}(\omega^2+\Delta)x(t)dt \quad \text{on } \Omega \\ \dfrac{\partial w}{\partial n} = -\dfrac{1}{\alpha}T^*\displaystyle\int_0^T e^{i\omega t}x(t)dt \quad \text{on } S \\ w = g \quad \text{on } \Gamma. \end{cases} \tag{7.8}$$

As easily seen, $A_\alpha : C([0,T]; H^2(\Omega)) \to H^2(\Omega)$ and it is linear. The representation result is the following

Theorem 7.5. *The minimum y_α^* is given by*

$$y_\alpha^* = A_\alpha P^\alpha \tag{7.9}$$

and it belongs to $H^2(\Omega)$, where (E^α, P^α) is the unique solution of the optimality system

$$\begin{cases} E_{tt} - \Delta E - B.A_\alpha P = 0 \\ P_{tt} - \Delta P - E\ = z \\ E(0) = E_t(0) = 0 \\ P(T) = P_t(T) = 0 \\ E = P = 0 \quad \text{ on } \Gamma \\ \dfrac{\partial E}{\partial n} = \left(TA_\alpha P - \dfrac{\partial A_\alpha P}{\partial n}\right)e^{-i\omega t} \ \text{ on } S \\ \dfrac{\partial P}{\partial n} = 0 \ \text{ on } S \end{cases} \tag{7.10}$$

Proof. For $y \in H^1$, the differential of J_α in y along $R \in H^1_\Gamma(\Omega)$ takes the form

$$D_y J_\alpha(y) \cdot R = \Re e \int_0^T \int_\Omega \left(\overline{E(t;y)(x) + z}\right)\eta(t)R dx dt \\ + \alpha \Re e \int_\Omega (\overline{y}R + \nabla\overline{y}\cdot\nabla R)dx \tag{7.11}$$

Since $P_{tt} - \Delta P = E + z$, the firs term on the right hand side of (7.11) becomes

$$\begin{aligned}
&\int_0^T \int_\Omega \left(\overline{P_{tt} - \Delta P}\right) \eta(t) R dx dt = \\
&\quad = \int_0^T \int_\Omega \overline{P(t;y)} \left((\eta(t)R)_{tt} - \Delta\eta(t)R\right) dx dt \\
&\quad + \int_\Omega \overline{P_t(T;y)}\eta(T)Rdx - \int_\Omega \overline{P_t(0;y)}\eta(0)Rdx \\
&\quad - \int_\Omega \overline{P(T;y)}\eta_t(T)Rdx + \int_\Omega \overline{P(0;y)}\eta_t(0)Rdx \\
&\quad - \int_0^T \int_{\partial\Omega} \left(\frac{\partial \overline{P(t;y)}}{\partial n} \eta(t) R - \overline{P(t;y)} \frac{\partial \eta(t) R}{\partial n} \right) ds(x) dt \\
&\quad = \int_0^T \int_\Omega \overline{P(t;y)} B(t) R dx dt + \int_0^T \int_S \overline{P(t;y)} \frac{\partial \eta(t) R}{\partial n} ds(x) dt.
\end{aligned} \tag{7.12}$$

Here we have integrated by parts and used the boundary conditions of η and P. By using formula (7.6) and still remembering the boundary conditions for P and η, we find that (7.12) is equal to

$$(7.12) = \int_0^T \int_\Omega \overline{e^{i\omega t}(\omega^2 + \Delta)P(t;y)} R dx dt + \int_0^T \int_S \overline{e^{i\omega t} P(t;y)} T R dx dt.$$

Notice that the integration by parts takes sense since $P(t;y) \in H^2(\Omega)$.

Then, the differential of J_α takes the form

$$\begin{aligned}
D_y J_\alpha(y) \cdot R = &\Re e \int_0^T \int_\Omega \overline{e^{i\omega t}(\omega^2 + \Delta)P(t;y)} R dx dt \\
&+ \Re e \int_0^T \int_S \overline{e^{i\omega t} P(t;y)} T R ds(x) dt \\
&+ \int_\Omega (\overline{y} R + \nabla \overline{y} \cdot \nabla R) dx
\end{aligned} \tag{7.13}$$

Since y_α^* is the minimum, the differential of J_α at y_α^* has to satisfy the condition

$$D_y J_\alpha(y_\alpha^*) \cdot R = 0, \tag{7.14}$$

for any $R \in H_\Gamma^1(\Omega)$. Therefore, by (7.13), (7.14) we find that y_α^* is the minimum if and only if it holds

$$\Re e \int_\Omega \left(\int_0^T \overline{e^{i\omega t}(\omega^2 + \Delta)P(t;y_\alpha^*)} dt + \alpha(I - \Delta)\overline{y_\alpha^*} \right) R dx$$

$$+\int_S \left(\alpha \frac{\partial \overline{y^*_\alpha}}{\partial n} + \overline{T^*} \int_0^T \overline{e^{i\omega t} P(t; y^*_\alpha)} dt \right) R ds(x) = 0$$

for any $R \in H^1_\Gamma(\Omega)$. Then, y^*_α is the variational solution of the problem

$$\begin{cases} (I - \Delta) y^*_\alpha = -\dfrac{1}{\alpha} \displaystyle\int_0^T e^{i\omega t} (\omega^2 + \Delta) P(t; y^*_\alpha) dt \quad \text{on } \Omega \\ \dfrac{\partial y^*_\alpha}{\partial n} = -\dfrac{1}{\alpha} T^* \displaystyle\int_0^T e^{i\omega t} P(t; y^*_\alpha) dt \quad \text{on } S \\ y = g \quad \text{on } \Gamma \end{cases}$$

Thanks to the regularity of the terms, standard results show that $y \in H^2(\Omega)$. This proved the sufficient condition.

The necessary condition follows by a backward argument. □

Corollary 7.6. *If $y \in H^2(\Omega)$, for any $\alpha \geq 0$, the Fréchet differential of J_α at y belongs to $(H^1_\Gamma(\Omega))^*$ and has the representation*

$$\langle D_y J_\alpha(y), f \rangle_{\sigma((H^1_\Gamma(\Omega))^*, H^1_\Gamma(\Omega))} = \int_\Omega G_{\Omega,\alpha}(y)(\xi) f(\xi) d\xi + \int_S H_{S,\alpha}(y)(\xi) (\gamma_S f)(\xi) d\xi,$$

where $\langle \cdot, \cdot \rangle_{\sigma((H^1_\Gamma(\Omega))^*, H^1_\Gamma(\Omega))}$ is the duality between $(H^1_\Gamma(\Omega))^*$ and $H^1_\Gamma(\Omega)$, $\gamma_S \in \mathcal{L}(H^1(\Omega); H^{1/2}(S))$ is the trace operator and

- *$G_{\Omega,\alpha}(y)$ belongs $L^2(\Omega)$ and it is given by*

$$G_{\Omega,\alpha}(y)(\xi) = \Re e \left(\int_0^T \overline{e^{i\omega t} (\omega^2 + \Delta) P(t; y)} dt \right)(\xi) + \Re e\, \alpha ((I - \Delta) \overline{y})(\xi), \quad \xi \in \Omega$$

- *$H_{S,\alpha}(y)$ belongs to $H^{1/2}(S)$ and it is given by*

$$H_{S,\alpha}(y)(\xi) = \Re e \left(T \left(\int_0^T \overline{e^{i\omega t} P(t; y)} dt \right)(\xi) + \alpha \frac{\partial \overline{y}}{\partial n}(\xi) \right), \quad \xi \in S$$

Proof. Take formula (7.13). By integration by part (which holds by the fact that $y \in H^2(\Omega)$) we can write

$$\begin{aligned} D_y J_\alpha(y) \cdot R =& \Re e \int_0^T \int_\Omega \overline{e^{i\omega t} (\omega^2 + \Delta) P(t; y)} R dx dt \\ &+ \Re e \int_0^T \int_S \overline{e^{i\omega t} P(t; y)} T R ds(x) dt \\ &+ \Re e\, \alpha \int_\Omega (I - \Delta) \overline{y} R dx \\ &+ \Re e\, \alpha \int_S \frac{\partial \overline{y}}{\partial n} R ds(x). \end{aligned}$$

Since $P \in C([0,T], H^2(\Omega))$ we have that the integral on Ω can be represented as a scalar product on $L^2(\Omega)$ between R and $G_{\Omega,\alpha}(y)$. On the other side, since $y \in H^2(\Omega)$ we have $\partial y/\partial n \in H^{-1/2}(S)$. Moreover,

$$\int_0^T \int_S \overline{e^{i\omega t} P(t;y)} T R ds(x) dt = \int_S T \left(\int_0^T \overline{e^{i\omega t} P(t;y)} dt \right) R ds(x)$$

Here we have used $\overline{T^*} = T$ and the fact that $T \in \mathcal{L}(H^{1/2}(S); H^{-1/2}(S))$.

Then,

$$T \left(\int_0^T \overline{e^{i\omega t} P(t;y)} dt \right) + \alpha \frac{\partial \overline{y}}{\partial n}$$

can be viewed as an element of $H^{-1/2}(S)$. This concludes the proof. □

Theorem 7.7. *There exists a unique y^* solving*

$$G_{\Omega,0}(y) + \gamma_S^* H_{S,0}(y) = 0,$$

which coincide with the harmonic state. Proof: Since the functional $J(y)$ has a unique minimum given by the harmonic state, say y^*, it follow that its Fréchet differential at this point vanishes. Hence, by Corollary 7.6, we obtain the result.

Next we have a result of convergence,.

Theorem 7.8. *Let $\alpha > 0$ and let (P^α, E^α) be the solution of equation* (7.10)*. Then $E^\alpha(t) = E(t; y_\alpha^*)$, $P^\alpha(t) = P(t; y_\alpha^*)$.*

Theorem 7.9. *We have*

$$\lim_{\alpha \to 0^+} y_\alpha^* = y^*$$

weakly in $H^1_\Gamma(\Omega)$, where y^* is the harmonic solution of problem (6.1)*.*

Proof. Let y^* be the unique solution of the Helmholtz equation

$$\begin{cases} k^2 y + \Delta y = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ y = g, \quad \text{on } \Gamma, \\ \displaystyle\lim_{r\to\infty} \int_{\partial B_r} \left| \frac{\partial y}{\partial r} - iky \right|^2 ds(x) = 0. \end{cases} \tag{7.15}$$

Has well known, the restriction of y^* to Ω, still denoted by y^*, give the harmonic solution of problem (6.1). That is, if $y = y^*$ in the initial conditions of (6.1), we have $\mathcal{E}(t; y^*) = y^* e^{-i\omega t}$, $t \geq 0$. In particular, $y^* = g$ on Γ, $\partial y^*/\partial n = T y^* \in H^{-1/2}(S)$ (by the definition of the operator T) and it is analytic, then $y^* \in K_g$. By (7.5) we also have $E(t; y^*) + z = 0$. Therefore

$$0 \leq J_\alpha(y_\alpha^*) \leq J_\alpha(y^*) = \frac{\alpha}{2} \left(\int_\Omega |y^*|^2 dx + \int_\Omega |\nabla y^*|^2 dx \right). \tag{7.16}$$

This implies $\|y^*_\alpha\|_{H^1} \leq \|y^*\|_{H^1}$, $\forall \alpha \geq 0$. Consequently, there exists a subsequence $y^*_{\alpha_n}$, $n \in \mathbb{N}$ and a function $\hat{y} \in H^1(\Omega)$ such that $\alpha_n \searrow 0$ and $y^*_{\alpha_n} \to \hat{y}$ weakly in $H^1(\Omega)$.

We now show that $\hat{y} = y^*$.

By taking into account that $y \mapsto E(t;y)$ is linear and that $y \mapsto E(t;y) + z = \mathcal{E}(t;y) - ye^{-i\omega t}$ is continuous with respect to the norm topology of $H^1(\Omega)$, it follows that the function

$$K_g \to \mathbb{R}; \quad y \mapsto \int_0^T \int_\Omega |E^\alpha(t;y)(x) + z|^2 \, dxdt$$

is convex. Therefore, it is lower semi continuous with respect to the weak topology of $H^1(\Omega)$. Then it holds

$$\begin{aligned}
\int_0^T \int_\Omega |E^\alpha(t;\hat{y})(x) + z|^2 \, dxdt &\leq \lim_{n\to\infty} \int_0^T \int_\Omega \left|E^{\alpha_n}(t; y^*_{\alpha_n}) + z\right|^2 dxdt \\
&\leq \lim_{n\to\infty} J_{\alpha_n}(y^*_{\alpha_n}) \\
&\leq \lim_{n\to\infty} J_{\alpha_n}(y^*) \\
&= \lim_{n\to\infty} \frac{\alpha}{2}\left(\int_\Omega |y^*|^2 dx + \int_\Omega |\nabla y^*|^2 dx\right) = 0.
\end{aligned}$$

This implies $E(t;\hat{y}) + z(t) = \mathcal{E}(t;\hat{y}) - \hat{y}e^{-i\omega t} = 0$, for almost all $t \in [0,T]$. By the continuity of $\hat{y} \mapsto \mathcal{E}(t;\hat{y}) + z(t)$ with respect to t, it follows $\mathcal{E}(t;\hat{y}) = \hat{y}e^{-i\omega t}$ for all $t \in [0,T]$. Since the harmonic solution is unique, it follows that $\hat{y} = y^*$. Then, $y^*_\alpha \to y^*$ weakly as $\alpha \to 0$. $\square$

8 The controlled system

8.1 Setting of the problem

Let us consider the following controlled system

$$\begin{cases}
\dfrac{\partial^2}{\partial t^2}\mathcal{E} - \Delta\mathcal{E} = Fe^{-i\omega t}, & \text{in } [0,\tau]\times\Omega, \\
\mathcal{E}|_\Gamma(t) = g(t) & t \in [0,\tau] \\
\frac{\partial \mathcal{E}(t)}{\partial n} = T_S u(t), & t \in [0,\tau] \\
\mathcal{E}(0) = y_1 & \\
\mathcal{E}_t(0) = y_2. &
\end{cases}$$

where $F \in H^1(\Omega)$, $g \in H^1([0,\tau]; H^{1/2}(\Gamma))$, $y_1, y_2 \in H^1(\Omega)$ with $y_1|_\Gamma = g$, $y_2|_\Gamma = g'$. The control, u, acts on S and it is taken in the space $L^2([0,\tau]; H^{1/2}(S))$.

Our aim is to find u such that, after a fixed time $\varepsilon \in (0,\tau)$, leads the solution $\mathcal{E}$ of the system to have a harmonic regime in a region $\Omega' \subset\subset \Omega$. That is, we search for a control u such that

$$\mathcal{E}(t,x) = y^*(x)e^{-i\omega t}, \quad (t,x) \in [\varepsilon,\tau]\times\Omega'.$$

Now, take $\rho \in C\infty([0,\tau] \times \overline{\Omega}; [0,1])$ with compact support in $(0,\tau] \times \Omega$ and such that $\rho = 1$ on $[\varepsilon, \tau] \times \Omega'$. It is clear that such a control, say u_ρ^*, minimizes the functional

$$J_\rho(u_\rho^*) = \min_{u \in L^2([0,\tau];H^{1/2}(S))} J_\rho(u) = \frac{1}{2}\int_0^\tau \int_\Omega \rho(t,x) \left| \frac{\partial}{\partial t}\mathcal{E}(t,x) + i\omega\mathcal{E}(t,x) \right|^2 dt$$

8.2 The control functional

In order to proceed, we recall the duality of the boundary space $H^{1/2}(S)$. We remember that $H^{1/2}(S)$ consists in the trace on S of all the functions of $H^1(\Omega)$. We endowe $H^{1/2}(S)$ with the following norm

$$\|u\|_{H^{1/2}(S)} = \sqrt{\int_S Nu(x)u(x)d\sigma(x)},$$

where $N : H^{1/2}(S) \to H^{-1/2}(S)$ is the Dirichlet to Neumann map defined as

$$Nu = \frac{\partial Y}{\partial n} \quad \text{on } S,$$

where Y is solution of the Laplace equation on Ω with Dirichlet boundary condition equal zero on Γ and equal u on S, that is Y solves

$$\begin{cases} \Delta Y = 0, & \text{on } \Omega \\ Y = 0, & \text{on } \Gamma \\ Y = u, & \text{on } S. \end{cases} \tag{8.1}$$

One can see that the norm on $H^{1/2}(S)$ is defined through the duality between $H^{1/2}(S)$ and its topological dual $H^{-1/2}(S)$, i.e.

$$\int_S Nu(x)u(x)d\sigma(x) = \langle Nu, u\rangle_{\sigma(H^{-1/2}(S),H^{1/2}(S))}.$$

In this case Nu is a function, which belongs to $H^{-1/2}(S)$ and then the duality can be expressed as an integral on S.

In order to verify that $\|\cdot\|_{H^{1/2}(S)}$ defines a norm, it is sufficient to observe that the Gauss formula leads to

$$\int_S Nu(x)u(x)d\sigma(x) = \int_\Omega |\nabla Y(x)|^2 dx$$

which is, of course, a norm on $H^{1/2}(S)$.

It worth to notice that N is invertible. In particular,

$$N^{-1} : H^{-1/2}(S) \to H^{1/2}(S))$$

and $N^{-1}h$, $h \in H^{-1/2}(S)$ correspond to the trace on S of the solution of the Laplace equation (8.1) with Neumann boundary condition on S equal to h.

8.3 The approximated functional

We consider here a perturbation of the functional J_ρ. The idea is that we are able to give an explicit expression for the optimal control of the approximated problem, and then we show that these optimal controls converge to the optimal control of J_ρ as the perturbation vanishes.

We want to minimize the functional

$$J_{\rho,\alpha}(u) = J_\rho(u) + \frac{\alpha}{2}\int_0^\tau \|u(t)\|^2_{H^{1/2}(S)} dt$$

over all $u \in L^2([0,\tau]; H^{1/2}(S))$.

In order to study the functional, we need to consider an order reduction of the equations. We consider the matrix A defined as follows:

$$\forall(\theta,\Psi) \in (L^2(\Omega) \times H^1_\Gamma(\Omega))^2$$

$$< A.\theta, \Psi >_{\sigma(L^2\times H^{-1}_{\bar\Omega},\, L^2\times H^1_\Gamma)} = \int_\Omega (\theta_2\Psi_1 - \nabla\theta_1.\nabla\Psi_2)dx$$

And defined by $\mathcal{B}_S = (0, B)$ with $B \in \mathcal{L}(H^{1/2}(S), H^{-1/2}(S))$

$$< \mathcal{B}_S u, \Psi >= \int_S \mathcal{T}_S u\ \Psi_2\, ds$$

Clearly, E solves

$$\begin{cases} \dfrac{\partial}{\partial t}E - AE = \tilde{F} + \mathcal{B}_S u, & \text{in } [0,\tau]\times\Omega, \\ E(t)|_\Gamma \cdot \begin{pmatrix}1\\0\end{pmatrix} = g(t), & t \in [0,\tau] \\ \\ E(0) = \begin{pmatrix}y_1\\y_2\end{pmatrix}. \end{cases} \tag{8.2}$$

We denote by η_R the the derivative of $\mathcal{E}$ with respect to the control u along the direction $R \in L^2([0,\tau]; H^{1/2}(S))$. Setting $H_R = \binom{\eta_R}{\partial\eta_R/\partial t}$, then H_R solves solves

$$\begin{cases} \dfrac{\partial}{\partial t}H_R(t) - AH_R = \mathcal{B}_S R, & \text{in } [0,\tau]\times\Omega, \\ H(t)|_\Gamma \cdot \begin{pmatrix}1\\0\end{pmatrix} = 0,\ t \in [0,T] \\ \\ H_R(0) = \begin{pmatrix}0\\0\end{pmatrix}. \end{cases}$$

With these notations the functional $J_{\rho,\alpha}$ takes the form

$$J_{\rho,\alpha}(u) = \frac{1}{2}\int_0^\tau \left(\rho(t)\int_\Omega \left(\begin{pmatrix} i\omega \\ 1\end{pmatrix}\cdot E(t)\right)^2 dx\right) dt \\ + \frac{\alpha}{2}\int_0^\tau \langle Nu, u\rangle_{\sigma(H^{-1/2}(S),H^{1/2}(S))} dt$$

Theorem 8.1. *For any $\alpha > 0$, $u \in L^2([0,\tau]; H^{1/2}(S))$ the functional $J_{\rho,\alpha}$ is differentiable in u, and its differential takes the form*

$$DJ_{\rho,\alpha}(u) = \Re e\, T_S^* P_2 + \alpha\, Nu \tag{8.3}$$

where $P_2 = P \cdot \binom{0}{1}$ and P solves the adjoint equation

$$\begin{cases} \dfrac{\partial}{\partial t}P + A^T P = \rho \begin{pmatrix} \omega^2 & -i\omega \\ i\omega & 1\end{pmatrix} E, & \text{in } [0,\tau]\times\Omega, \\ P(t)|_\Gamma = 0, & t \in [0,\tau] \\ \dfrac{\partial P(t)}{\partial n} = 0, & t\in[0,\tau] \\ P(T) = \begin{pmatrix} 0 \\ 0\end{pmatrix}. \end{cases} \tag{8.4}$$

Remark 8.2. The expression in (8.3) means that for any $R \in L^2([0,\tau]; H^{1/2}(S))$ we have

$$\langle DJ_{\rho,\alpha}(u), R\rangle_{\sigma(H^{-1/2}(S),H^{1/2}(S))} = -\Re e \int_0^\tau \int_S ((T_S^* P_2)\,\overline{R}) d\sigma(x) dt \\ + \alpha \int_0^\tau \langle Nu(t), R(t)\rangle_{\sigma(H^{-1/2}(S),H^{-1/2}(S))} dt.$$

In the proof of Theorem 8.1 the duality $\langle\cdot,\cdot\rangle_{\sigma(H^{-1/2}(S),H^{1/2}(S))}$ will be replaced by the dot "$\cdot$" in order to avoid heavy notations.

Proof of Theorem (8.1). Fix $\alpha > 0$ and $u \in L^2([0,\tau]; H^{1/2}(S))$. The differential of the J_α in u along $R \in L^2([0,\tau]; H^{1/2}(S))$ takes the form

$$\begin{aligned} DJ_{\rho,\alpha}(u)\cdot R =& \Re e \int_0^\tau \rho(t)\int_\Omega \left(\begin{pmatrix} i\omega \\ 1\end{pmatrix}\cdot E(t)\right)\left(\overline{\begin{pmatrix} i\omega \\ 1\end{pmatrix}\cdot H_R(t)}\right) dt \\ &+ \alpha \int_0^\tau \langle Nu(t), R(t)\rangle_{\sigma(H^{-1/2}(S),H^{-1/2}(S))} dt \end{aligned} \tag{8.5}$$

Let P be the solution of the adjoint system (8.4). The right-hand term in (8.4) follows by the equality

$$\rho\left(\begin{pmatrix} i\omega \\ 1\end{pmatrix}\cdot E\right)\begin{pmatrix} -i\omega \\ 1\end{pmatrix} = \begin{pmatrix} \omega^2 & -i\omega \\ i\omega & 1\end{pmatrix} E.$$

By taking into account the regularity of E, standard results show that $P = \binom{P_1}{P_2}$ belongs to $C([0,\tau]; H^1_\Gamma(\Omega)) \times C^1([0,\tau]; L^2(\Omega))$. A straightforward computation shows

$$\begin{aligned}
&\int_0^\tau \int_\Omega \left(\frac{\partial P}{\partial t} + A^T P\right) \cdot \left(\overline{H_R(t)}\right) dt \\
&= \int_0^\tau \int_\Omega P(t) \cdot \left(\overline{-\frac{\partial H_R}{\partial t} + AH_R}\right) dxdt + \int_0^\tau \int_{\partial\Omega} \left(\frac{\partial P_2}{\partial n}\overline{\eta_R} - P_2 \frac{\partial \overline{\eta_R}}{\partial n}\right) d\sigma(x)dt \\
&= -\int_0^\tau \int_S P_2\,(\overline{T_S R}) d\sigma(x)dt \\
&= -\int_0^\tau \int_S (T_S^* P_2)\,\overline{R} d\sigma(x)dt.
\end{aligned}$$

Then

$$\begin{aligned}
DJ_{\rho,\alpha}(u) \cdot R = -\Re e \int_0^\tau \int_S ((T_S^* P_2)\,\overline{R}) d\sigma(x)dt \\
+ \alpha \int_0^\tau \langle Nu(t), R(t)\rangle_{\sigma(H^{-1/2}(S),H^{-1/2}(S))} dt.
\end{aligned}$$

Notice that R is real, then $\overline{R} = R$. Therefore, (8.3) follows. $\square$

Corollary 8.3. *The minimum of $J_{\rho,\alpha}$ is reached in*

$$u^*_{\rho,\alpha} = \frac{1}{\alpha} N^{-1} \left(\Re e\, T_S^* P_2\right),$$

where $P_2 = P \cdot \binom{0}{1}$ and P solves (8.4). That is,

$$J(u^*_{\rho,\alpha}) = \min_{u \in L^2([0,\tau]; H^{1/2}(S))} J_{\rho,\alpha}(u).$$

8.4 Riccati equation

In this section we investigate the use of the backward Riccati equation of classical "linear-quadratic" optimal control theory in order to avoid the backward calculus of the adjoint state equation P.

In what follows, P, E are the solutions of state and co-state equation respectively. We also write

$$\tilde{F}(t) = \begin{pmatrix} 0 \\ Fe^{-i\omega t} \end{pmatrix}, \quad \tilde{\omega} = \begin{pmatrix} \omega^2 & -i\omega \\ i\omega & 1 \end{pmatrix}.$$

We want to write P in the form

$$P(t) = Q(t)E(t) + \theta(t), \quad t \in [0, T],$$

where Q is a 2×2 matrix depending on $t \in [0,T]$, $\theta \in C([0,T],\mathbb{R}^2)$.

A formal calculus shows

$$\frac{\partial}{\partial t}(QE+\theta) = Q_tE + QE_t + \theta_t = Q_tE + QAE + Q\mathcal{B}_S u + Q\tilde{F} + \theta_t.$$

On the other side, $QE + \theta = P$, then

$$\frac{\partial}{\partial t}P = -A^T P + \rho\tilde{\omega}E = -A^T(QE+\theta) + \rho\tilde{\omega}E.$$

Therefore, Q, θ have to satisfy the equation

$$-A^TQE - A^T\theta + \rho\tilde{\omega}E = Q_tE + QAE + Q\mathcal{B}_S u + Q\tilde{F} + \theta_t,$$

that is as, $Q\mathcal{B}_S u = 1/\alpha\ QN^{-1}(\mathcal{R}e\mathcal{T}_S^* P_2) = QN^{-1}(\mathcal{R}e\mathcal{T}_S^*(QE+\theta_2)$,

$$(Q_t + QA + A^TQ - \tilde{\omega})E = 1/\alpha\ QN^{-1}(\mathcal{R}e\mathcal{T}_S^*(QE+\theta_2)) + \tilde{F} + \theta_t + A^T\theta.$$

Since we want that this equality holds for any solution E of equation (8.2), to reach the result it is sufficient to find Q, θ solving the Riccati equation

$$\begin{cases} Q_t + QA + A^TQ = \rho\tilde{\omega}, & \text{in } [0,\tau], \\ \theta_t + A^T\theta + Q\tilde{F} = 0, & \text{in } [0,\tau], \\ Q(\tau) = 0, \\ \theta(\tau) = 0. \end{cases} \tag{8.6}$$

There is a huge literature about Riccati equations (see, for instance, [1]), which tell us that

Theorem 8.4. *There exists a unique* $Q \in C([0,\tau]\times\Omega;\mathbb{R}^4)$ *and* $\theta \in C([0,\tau)\times\Omega;\mathbb{R}^2)$ *which solve equation* (8.6).

We are going to give a more explicit expression for the Riccati equations solved by Q. Write

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$$

We have

$$QA = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}\begin{pmatrix} 0 & \Delta \\ I & 0 \end{pmatrix} = \begin{pmatrix} Q_2 & Q_1\Delta \\ Q_4 & Q_3\Delta \end{pmatrix}$$

$$A^TQ = \begin{pmatrix} 0 & \Delta \\ I & 0 \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} = \begin{pmatrix} \Delta Q_3 & \Delta Q_4 \\ Q_1 & Q_2 \end{pmatrix}$$

Then, taking into account (8.6), $Q_1, \ldots, Q_4$ have to satisfy

$$\frac{\partial Q_1}{\partial t} + Q_2 + \Delta Q_3 = \rho\omega^2;$$

$$\frac{\partial Q_2}{\partial t} + Q_1\Delta + \Delta Q_4 + = i\rho\omega$$
$$\frac{\partial Q_3}{\partial t} + Q_4 + Q_1 = i\rho\omega$$
$$\frac{\partial Q_4}{\partial t} + Q_3\Delta + Q_2 = \rho$$

With the final conditions $Q_i(T) = 0$, $i = 1, \ldots, 4$.

We can now give an explicit expression to the optimal control u^*_α which minimizes the functional J_α. We have

$$u^*_\alpha(t) = \frac{1}{\alpha} N^{-1} \left(\Re e \left(Q(t)E(t) + \theta(t)) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right)$$
$$= \frac{1}{\alpha} N^{-1} \left(\Re e \left(Q_3(t)\mathcal{E}(t) + Q_4(t)\frac{\partial \mathcal{E}}{\partial t}(t) + \theta_2(t) \right) \right).$$

8.5 Optimality system

We consider the unique solution $\mathcal{E}^\alpha$ to the following evolution optimality; this solution solves the out going radiating condition (4.1):

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\mathcal{E}^\alpha - \Delta\mathcal{E}^\alpha = Fe^{-i\omega t}, & \text{in } [0,\tau]\times\Omega, \\ \mathcal{E}^\alpha|_\Gamma(t) = g(t) & t \in [0,\tau] \\ \frac{\partial \mathcal{E}^\alpha(t)}{\partial n} = \frac{1}{\alpha} T_S.N^{-1}\left(\Re e\left(Q_3(t)\mathcal{E}^\alpha(t) + Q_4(t)\frac{\partial \mathcal{E}^\alpha}{\partial t}(t) + \theta_2(t)\right)\right), & t \in [0,\tau] \\ \mathcal{E}^\alpha(0) = y_1 \\ \mathcal{E}^\alpha_t(0) = y_2. \end{cases}$$

Proposition 8.5. *As $\alpha \to 0$, $\alpha > 0$ we have*

$$\|\frac{\partial}{\partial t}\mathcal{E}^\alpha + i\,\omega\mathcal{E}^\alpha\|_{L^2(\epsilon,1,L^2(\Omega))} \longrightarrow 0$$

proof:
$$\int_\epsilon^\tau \int_\Omega |\frac{\partial}{\partial t}\mathcal{E}^\alpha + i\,\omega\mathcal{E}^\alpha|^2 dt dx \leq J_{\alpha_n}(u^*_{\alpha_n}) \leq J_{\alpha_n}(u^*)$$
$$= \alpha_n \int_0^\tau < N.u^*, u^* > dt \to 0,\ n \to \infty$$

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